LIMIT PROBABILITIES FOR CRITICAL AGE-DEPENDENT BRANCHING PROCESSES WITH IMMIGRATION

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Limit Probabilities for Critical Age-Dependent Branching Processes with Immigration

by

Howard J. Weiner

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1. Introduction.

(1.1) Let $Z(t)$ denote the number of cells alive at time t in a standard critical age-dependent branching process ([1], Chapter 4) with absolutely continuous cell lifetime distribution function

(1.2) $$G(t)\ ,\quad G(0+) = 0$$

and satisfying

(1.3) $$0 < \mu \equiv \int_0^\infty t dG(t)\ .$$

Let

(1.4) $$g(t) \equiv G'(t)$$

be the density of G. Assume

(1.5) $$\int_0^\infty t^{b+4} g(t)\ dt < \infty$$

with b given by (1.15).

At the end of each cell life, the original cell disappears, and is replaced by k new cells with probability $p_k \geq 0$ and

$$\sum_{k=0}^{\infty} p_k = 1 , \tag{1.6}$$

satisfying criticality

$$\sum_{k=1}^{\infty} kp_k = 1 . \tag{1.7}$$

Let, for $0 \leq s \leq 1$

$$h(s) \equiv \sum_{k=0}^{\infty} p_k s^k \tag{1.8}$$

and assume that, for some $\epsilon > 0$,

$$h(1+\epsilon) \text{ exists} . \tag{1.9}$$

This guarantees, in particular, that for $n \geq 1$,

$$\sum_{k=1}^{\infty} k^n p_k \text{ exists} \tag{1.10}$$

and that all derivatives of $h(s)$ for $0 \leq s \leq 1$ exist at $s = 1$ and can be evaluated by interchanging derivatives and summation.

Assume in addition that

$$0 < h''(1) . \tag{1.11}$$

(1.12) Let $N(t)$ denote the total progeny born by time t in a critical age-dependent process satisfying (1.1)-(1.11).

(1.13) Let $Z_0(t)$ denote the number of cells alive at t in a cell immigration process in which new-born cells are introduced at renewal epochs. The (random) time between epochs is governed by a continuous distribution function $G_0(t)$, $G_0(0+) = 0$

with

(1.14) $$0 < \mu_0 \equiv \int_0^\infty t dG_0(t)$$

and for

(1.15) $$b \equiv \frac{2\mu m_0}{\mu_0 h''(1)} \quad \text{(with } m_0 \text{ defined below)}$$

that, as $t \to \infty$,

(1.16) $$t^{b+2}(1-G_0(t)) \to 0 \ .$$

At each renewal epoch, k new cells are introduced with probability p_{0k} and let, for $0 \leq s \leq 1+\epsilon$ for some $\epsilon > 0$

(1.17) $$h_0(s) \equiv \sum_{k=0}^{\infty} p_{0k} s^k < \infty$$

and

(1.18) $$0 < m_0 = h_0'(1)$$

and

$$h_0''(1) < \infty \ , \quad h''(1) < \infty \ .$$

Each new cell introduced at a renewal epoch now is part of the process and initiates, independent of all other cells and the immigration process, a critical age-dependent branching process satisfying (1.1)-(1.11).

(1.19) Let $N_0(t)$ denote the total progeny by time t of the immigration process satisfying (1.1)-(1.18).

It is the purpose of this paper to show that for $k \geq 1$, as $t \to \infty$,

$$P_{0k}(t) = P[Z_0(t)=k] \sim \frac{c}{t^b} \tag{1.20}$$

where

$$b = \frac{2\mu m_0}{\mu_0 h''(1)} \quad \text{and}$$

where $c > 0$ denotes a constant which may depend on k and under the additional hypotheses that

$$p_{0k} > 0 \quad \text{all} \quad k \geq 0 \ , \tag{1.21}$$

and that there is a unique $\alpha > 0$ defined by

$$p_{00} \int_0^\infty e^{\alpha y} dG_0(y) = 1 \tag{1.22}$$

that, as $t \to \infty$, for $k \geq 0$,

$$(1.23) \qquad Q_{0k}(t) = P[N_0(t)=k] \sim ce^{-\alpha t}$$

for c (depending on k) some positive constant. A multi-dimensional version and extension are indicated in Section 3.

2. Integral Equations.

For reference later, some results about $Z(t)$ are listed. See [1], Chapter 4, for example.

Let, for $0 \leq s \leq 1$

$$(2.1) \qquad E(s^{Z(t)}) \equiv F(s,t) .$$

Then, by notation (1.1)-(1.11)

$$(2.2) \qquad F(s,t) = s(1-G(t)) + \int_0^t h(F(s,t-u))dG(u) .$$

Under the hypotheses (1.1)-(1.11), denoting

$$(2.3) \qquad P_k(t) \equiv P[Z(t)=k] ,$$

then [3]

$$(2.4) \qquad P_1(t) = 1 - G(t) + \int_0^t h'(1-P(t-u))P_1(t-u)dG(u)$$

and in general, for $k \geq 2$,

$$(2.5) \qquad P_k(t) = f_k(t) + \int_0^t h'(1-P(t-u))P_k(t-u)dG(u) ,$$

where

(2.6) $$P(t) \equiv P[Z(t) > 0] .$$

By [1], [3] respectively,

(2.7) $$P(t) \sim (2\mu)(h''(1)t)^{-1}$$

and for $k \geq 1$,

(2.8) $$P_k(t) \sim \frac{c_k}{t^2} ,$$

where $c_k > 0$ is a constant, possibly depending on k.

Denote, for $0 \leq s \leq 1$,

(2.9) $$F(s,t) \equiv Es^{Z(t)} = \sum_{k=0}^{\infty} P[Z(t)=k]s^k .$$

(2.10) $$F_0(s,t) \equiv Es^{Z_0(t)} = \sum_{k=0}^{\infty} P[Z_0(t)=k]s^k .$$

(2.11) $$H(s,t) \equiv Es^{N(t)} = \sum_{k=1}^{\infty} P[N(t)=k]s^k .$$

(2.12) $$H_0(s,t) \equiv Es^{N_0(t)} = \sum_{k=0}^{\infty} P[N_0(t)=k]s^k .$$

Then the following theorem holds.

<u>Theorem 1</u>. <u>Assume</u> (1.1)-(1.18) <u>hold</u>. <u>Then</u> <u>for</u> $k \geq 0$, <u>as</u> $t \to \infty$,

(2.13) $$P[Z_0(t)=k] \sim \frac{c}{t^b}$$

where $c > 0$ depends on k.

Proof. By [2]

$$F_0(s,t) = 1 - G_0(t) + \int_0^t h_0(F(s,t-u))F_0(s,t-u)dG_0(u) \, . \tag{2.14}$$

For $\ell \geq 0$ an integer, denote by

$$P_{0\ell}(t) \equiv P[Z_0(t)=\ell] \tag{2.15}$$

$$P_\ell(t) \equiv P[z(t)=\ell] \tag{2.16}$$

and

$$P(t) = P[Z(t) > 0] \, . \tag{2.17}$$

From the assumptions we note that

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F_0(s,t)\Big|_{s=0} = P_{0\ell}(t) \tag{2.18}$$

and

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F(s,t)\Big|_{s=0} = P_\ell(t) \, . \tag{2.19}$$

By (2.18) applied to (2.14) for $\ell = 0$

$$P_{00}(t) = 1 - G_0(t) + \int_0^t h_0(1-P(t-u))P_{00}(t-u)dG_0(u) \, . \tag{2.20}$$

Define

(2.21) $$R(t) \equiv 1 - G_0(t) + \frac{1}{\mu_0}\int_0^t h_0(1-P(t-u))R(t-u)e^{-\frac{(t-u)}{\mu_0}}\,du$$

or equivalently,

$$R(t) = 1 - G_0(t) + \frac{e^{-\frac{t}{\mu_0}}}{\mu_0}\int_0^t h_0(1-P(u))R(u)e^{-\frac{u}{\mu_0}}\,du$$

Taking the derivative w.r.t. t in (2.21) and simplifying leads to the differential equation

(2.22) $$R'(t) + \frac{(1-h_0(1-P(t)))}{\mu_0}R(t) = f(t)$$

where

(2.23) $$f(t) = o(t^{-b-2}) .$$

Expanding $1-h_0(1-P(t))$ in a Taylor series, using (2.7) and the idea of the proof of Claim IV of ([3] pp 480-481), one may solve for $R(t)$ asymptotically to get

(2.24) $$R(t) \sim ct^{-b} , \quad \text{where} \quad c > 0$$

is a constant whose value may change from equation to equation. From (2.20), (2.21),

(2.25) $$P_{00}(t) - R(t) = \int_0^t h_0(1-P(t-u))(P_{00}(t-u)-R(t-u))dG_0(u)$$

$$+ \int_0^t h_0(1-P(t-u))R(t-u)(dG_0(u)-dE(u))$$

where

$$E(t) = 1 - e^{-\frac{u}{\mu_0}} . \tag{2.26}$$

Define

$$\Delta(t) = |P_{00}(t) - R(t)| . \tag{2.27}$$

Then, iterating (2.25) repeatedly, one obtains

$$\Delta(t) \leq \Delta \cdot G_{0n}(t) + R \cdot |G-E| \cdot U_0(t) \tag{2.28}$$

for all n, t, and the dots denote convolution integral, where $G_{0n}(t)$ is the n^{th} convolution of G_0 with itself, and

$$U_0(t) = \sum_{\ell=0}^{\infty} G_\ell(t) \sim \frac{t}{\mu_0} .$$

Let $n \to \infty$, then $t \to \infty$, and the law of large numbers and the properties of R, $G-E$, U_0 yield that

$$t^b \Delta(t) \to 0 \quad \text{as} \quad t \to \infty . \tag{2.29}$$

This yields the result of Theorem 1 for $P_{00}(t)$.

The argument for $P_{01}(t)$ is similar and uses the result for $P_{00}(t)$. The general result for $P_{0n}(t)$ follows by induction using Leibniz' rule for successive differentiation, and is omitted.

Remark: The proof of Theorem 1 of [3] on pp. 482-483 is incompletely justified and would go through by an argument as above.

Theorem 2. Assume (1.1)-(1.22) to hold. Then, for $k \geq 0$ *an integer*

$$Q_{0n}(t) \equiv P[N_0(t)=k] \sim ce^{-\alpha t} \tag{2.30}$$

for some $c > 0$ *depending on* k, *where* α *is as given in* (1.22).

Proof. By arguments similar to those used to establish (2.14) by the law of total probability,

$$H_0(s,t) = 1 - G_0(t) + \int_0^t h_0(H(s,t-u))H_0(s,t-u)dG_0(u) \ . \tag{2.31}$$

The assumptions of the theorem allow derivatives with respect to s to be taken under the summation sign in (2.11)-(2.12) and that for $\ell \geq 0$,

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} H(s,t)\Big|_{s=0} = P[N(t)=\ell] \equiv Q_\ell(t) \tag{2.32}$$

and

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} H_0(s,t)\Big|_{s=0} = P[N_0(t)=\ell] = Q_{0\ell}(t) \ , \tag{2.33}$$

and note that

$$Q_0(t) = P[N(t)=0] = 0 \ . \tag{2.34}$$

Applying (2.32)-(2.34) to (2.31) for $\ell = 0$ yields

$$Q_{00}(t) = 1 - G_0(t) + p_{00}\int_0^t Q_{00}(t-u)dG_0(u) \ . \tag{2.35}$$

But (2.35) is in the standard form of the integral equation for the mean number of cells at time t in a Bellman-Harris age-dependent branching process with cell lifetime distribution function $G_0(t)$ and mean number of progeny per parent of $0 < p_{00} < 1$, the subcritical case. (See [1] pp 162-168). Hence [1] as $t \to \infty$,

$$Q_{00}(t) \sim ce^{-\alpha t} , \tag{2.36}$$

where $c > 0$ may be explicitly evaluated [1], but since no general tractable expression for corresponding constants in the asymptotic form for $Q_{0\ell}(t)$ seems obtainable, such constants will not be evaluated explicitly, although this proof indicates how they may be obtained recursively.

Applying (2.32)-(2.34) to (2.31) for $\ell = 1$ yields

$$Q_{01}(t) = p_{01} \int_0^t Q_1(t-u)Q_{00}(t-u)dG_0(u) + p_{00} \int_0^t Q_{01}(t-u)dG_0(u) , \tag{2.37}$$

which can be expressed in the form

$$Q_{01}(t) = f(t) + p_{00} \int_0^t Q_{01}(t-u)dG_0(u) , \tag{2.38}$$

where, from [1] and (2.37), it follows that, as $t \to \infty$,

$$f(t) \sim ce^{-\alpha t} . \tag{2.39}$$

By Theorem 1 (i) of ([1] p. 145) and the argument of equation (9)-(11) on page 146 of [1], one then obtains

$$(2.40)\qquad Q_{01}(t) \sim ce^{-\alpha t} ,$$

for a $c > 0$ which may be evaluated, as indicated in the remark following (2.37).

The rest of the argument proceeds by induction analogous to that used in Theorem 1.

3. Multidimensional Case.

Let

(3.1) $Z_{ij}(t)$ = the number of cells of type j at time t starting with one new-born cell of type i at $t = 0$ with $1 \leq i \leq m$ in an m-type critical age-dependent branching process described as follows. At time $t = 0$, one newly born cell of type i starts the process, for some $1 \leq i \leq m$. The cell lives a random time described by a continuous distribution function

$$(3.2)\qquad G_i(t), \quad G_i(0+) = 0 .$$

At the end of its life, cell i is replaced by j_1 new daughter cells of type 1, j_2 new cells of type $2,\ldots,j_m$ cells of type m with probability $\mathcal{P}_{ij_1j_2j_3\cdots j_m}$.

Define the generating functions, for $\underline{s} = (s_1,\ldots,s_m)$, $\underline{j} = (j_1,\ldots,j_m)$, $\underline{s}^{\underline{j}} \equiv (s_1^{j_1},\ldots,s_m^{j_m})$.

$$(3.3)\qquad h_i(s_1,\ldots,s_m) \equiv h_i(\underline{s}) = \sum_{(j_1\cdots j_m)} \mathcal{P}_{ij_1\cdots j_m} s_1^{j_1}\cdots s_m^{j_m} \equiv \sum_{\underline{j}} p_{i\underline{j}}\underline{s}^{\underline{j}} .$$

Each daughter cell proceeds independently of the state of the system, with each cell type j governed by $G_j(t)$ and $h_j(\underline{s})$.

Assume, for $\underline{1+\epsilon} \equiv (1+\epsilon,\ldots,1+\epsilon)$ and $\underline{1} = (1,\ldots,1)$, m-vectors,

(3.4) $$h_i(\underline{1+\epsilon}) < \infty \quad \text{for} \quad 1 \leq i \leq m .$$

This insures that all moments of $h_i(\underline{s})$ evaluated at $\underline{s} = \underline{1}$ may be computed by partial differentiations under the summation sign.

Define, for $1 \leq i,\ j \leq m$,

(3.5) $$m_{ij} \equiv \left.\frac{\partial h_i(\underline{s})}{\partial s_j}\right|_{\underline{s}=\underline{1}} \equiv h_{ij}(\underline{1})$$

and assume

(3.6) $$m_{ij} > 0 \quad \text{all} \quad 1 \leq i, \quad j \leq m ,$$

and let the first moment $m \times m$ matrix be

(3.7) $$M = (m_{ij}) .$$

By standard Frobenius theory ([1], p. 185), there is a largest eigenvalue in absolute value, denoted ρ, which is positive.

<u>The basic assumption of criticality is that</u>

(3.7)(i) $$\rho = 1 .$$

It follows that there are strictly positive eigenvectors $\underline{u} > 0$, $\underline{v} > 0$ such that (see [4]), (3.7)(ii) $$M\underline{u} = \underline{u}\ , \quad \underline{v}M = \underline{v}\ ,$$

$$\sum_{i=1}^{m} u_i = 1 = \underline{u}\cdot\underline{1}\ ,$$

and

$$\underline{u}\cdot\underline{u} \equiv \sum_{\ell=1}^{m} u_\ell v_\ell = 1\ .$$

Assume

(3.7)(iii) $$\infty > \frac{\partial^2 h_i(\underline{1})}{\partial s_j \partial s_k} > 0\ , \quad 1 \le j,\ k \le m\ .$$

Denote

(3.7)(iv) $$Q(\underline{u}) \equiv \frac{1}{2} \sum_{i=1}^{m} \sum_{\ell=1}^{m} \sum_{r=1}^{m} \frac{\partial^2 h_i(\underline{1})}{\partial s_\ell \partial s_r} u_\ell u_r v_i < \infty\ ,$$

where, for $1 \le i \le m$, for $a > 0$ (3.9)

(3.8)(i) $$\int_0^\infty t^{a+4}\, dG_i(t) < \infty\ ,$$

and denote, $0 \le i \le m$

(3.8)(ii) $$0 < u_i \equiv \int_0^\infty t\, dG_i(t)\ ,$$

where $a > 0$ is given by

(3.9) $$a \equiv \frac{\left(\sum_{\ell=1}^{m} h_{0\ell}(\underline{1}) u_\ell\right)\left(\sum_{r=1}^{m} \mu_\ell u_\ell v_\ell\right)}{\mu_0 Q(\underline{u})}\ ,$$

with $h_{0\ell}(\underline{1}) \equiv \frac{\partial}{\partial s_\ell} h_0(\underline{1})$, assumed to exist.

Let

$$\underline{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{im}(t)) \; . \tag{3.10}$$

Let

$$\underline{N}_i(t) = (N_{i1}(t), N_{i2}(t), \ldots, N_{im}(t)) \tag{3.11}$$

denote the m-vector with entries

(3.12) $N_{ij}(t)$ = total progeny of type j born by t in the above critical m-type process starting with one new cell of type i.

An m-type branching process with immigration is defined as follows. At renewal epochs with inter-arrival time continuous distribution

$$G_0(t) \; , \tag{3.13}$$

$$G_0(0+) = 0, \; G_0(t) < 1 \quad \text{for all} \quad t > 0 \; , \tag{3.14}$$

satisfying

$$t^{4+a}(1-G_0(t)) \to 0 \quad \text{as} \quad t \to \infty \tag{3.15}$$

m-types of new cells are introduced such that there are i_1 new cells of type 1, i_2 new cells of type $2, \ldots, i_m$ cells of type m introduced with probability $p_{0i,\ldots,i_m}$. Denote

(3.16) $$h_0(\underline{s}) \equiv \sum_{(i_1,\ldots,i_m=0)}^{\infty} p_{0i_1\cdots i_m} s_1^{i_1}\cdots s_m^{i_m} \equiv \sum_{\underline{\ell}} p_{0\underline{\ell}}\underline{s}^{\underline{\ell}} ,$$

and assume

(3.17) $$h_0(\underline{1+\epsilon}) \text{ exists}$$

for some $\epsilon > 0$.

Each new cell of type i initiates an m-type critical age-dependent branching process [1] independent of all other cells and of the renewal process, satisfying (3.1)-(3.12).

Define, for $1 \leq i \leq m$,

(3.18) $$Z_{0i}(t) \text{ and } N_{0i}(t)$$

to be the number of cells of type i alive at t and the total progeny born by t, respectively, in the m-type branching process satisfying (3.1)-(3.17), called an m-type critical age-dependent branching process with immigration.

Denote

(3.19) $$\underline{Z}_0(t) \equiv (Z_{01}(t),Z_{02}(t),\ldots,Z_{0m}(t))$$

(3.20) $$\underline{N}_0(t) \equiv (N_{01}(t),N_{02}(t),\ldots,N_{0m}(t)) .$$

Theorem 3. *Under assumptions (3.1)-(3.12), for* $\underline{k} = (k_1,\ldots,k_m)$ *a vector of non-negative integers, at least one of which is strictly positive,*

$$(3.21) \qquad \lim_{t \to \infty} t^2 P[\underline{Z}_i(t)=\underline{k}] = c > 0$$

$$(3.22) \qquad \lim_{t \to \infty} P[\underline{N}_i(t)=\underline{k}] = d > 0$$

where c,d *are constants which may depend on* i,k.

Proof. The proof follows the one-dimensional case using [4] and is omitted.

Theorem 4. *Under assumptions* (3.11)-(3.20), *for* $\underline{\ell} = (\ell_1,\ldots,\ell_m)$ *a vector of non-negative integers,*

$$(3.23) \qquad \lim_{t \to \infty} t^a p[\underline{z}_0(t)=\underline{\ell}] = c > 0$$

for some constants c.

If

$$(3.24) \qquad p_{0\underline{\ell}} > 0$$

and there is a unique $\alpha > 0$ *defined by*

$$(3.25) \qquad h_0(0) \int_0^\infty e^{\alpha u} d\, G_0(u) = 1 \; ,$$

then

$$(3.26) \qquad \lim_{t \to \infty} e^{\alpha t} P[\underline{N}_0(t)=\underline{\ell}] = c > 0 \; .$$

Proof. Theorem 4 follows from Theorem 3 in a proof similar to Theorems 1 and 2, respectively.

Remark: If the quantities $\underline{Z}_i(t)$, $\underline{N}_i(t)$, $\underline{Z}_0(t)$, $\underline{N}_0(t)$, $\underline{k}$, $\underline{\ell}$ in Theorems 3 and 4 are replaced by corresponding marginal vectors of dimension $1 \leq d < m$, the corresponding results of Theorems 3 and 4 hold and are of the same form, since the method of proof is the same, with expressions of the same form.

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Let $Z_0(t)$, $N_0(t)$ denote, respectively, the number of cells alive at t and the total progeny born by t in a process with a random number of new cells introduced at renewal epochs, each new cell initiating a critical age-dependent branching process. As $t \to \infty$, the forms of $P[Z_0(t) = k]$ and $P[N_0(t) = k]$ are obtained for $k = 1,2,\ldots$ and $k = 0,1,2,\ldots,$ respectively. A multi-dimensional version and extension are indicated.

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